

# On a distribution function of a probability measure involving a permutation

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## Abstract

In [3], we have introduced a probability measure to study the power and exponential sums for a certain coding system. The distribution function of the probability measure gives explicit formulas for the power and exponential sums.

[3, Theorem 4] states that the higher order derivatives of the distribution function with respect to a certain parameter are expressed by a generalization of the Takagi function. In [3], we only gave the sketch of the proof of Theorem 4, because the complete proof is very long. The purpose of this paper is to give the complete proof of [3, Theorem 4].

## 1 Introduction

Let  $q \geq 2$  be an integer and  $\sigma$  be a permutation

$$\sigma = \begin{pmatrix} 0 & 1 & \cdots & q-1 \\ \sigma(0) & \sigma(1) & \cdots & \sigma(q-1) \end{pmatrix}.$$

Throughout the paper, we assume that  $\sigma^q = \text{id}$ . A probability measure involving  $\sigma$  on the unit interval has been introduced in [3]. Let us recall the definition of the measure.

Let  $I = I_0(0) = [0, 1]$ , and for each positive integer  $k$ , let

$$I_k(n) = \left[ \frac{n}{q^k}, \frac{n+1}{q^k} \right), \quad 0 \leq n \leq q^k - 2,$$

$$I_k(q^k - 1) = \left[ \frac{q^k - 1}{q^k}, 1 \right].$$

We denote the  $\sigma$ -field  $\sigma\{I_k(n); 0 \leq n \leq q^k - 1\}$  by  $\mathcal{F}_k$  and the  $\sigma$ -field  $\bigvee_{k=0}^{\infty} \mathcal{F}_k$  by  $\mathcal{F}$ .

**Definition 1.** Let  $\mathbf{d} = (d_0, \dots, d_{q-2})$  be a vector with  $0 < d_j < 1$  ( $0 \leq j \leq q-2$ ) and  $0 < \sum_{j=0}^{q-2} d_j < 1$ , and set  $d_{q-1} = 1 - \sum_{j=0}^{q-2} d_j$ . Let  $\mathbf{r} = (r_0, \dots, r_{q-2})$  be a vector whose components satisfy the same conditions as those of  $\mathbf{d}$ , and set  $r_{q-1} = 1 - \sum_{j=0}^{q-2} r_j$ . Then the probability measure  $\mu_{\mathbf{d}, \mathbf{r}}$  involving a permutation  $\sigma$  on  $(I, \mathcal{F})$  is defined as follows.

- (i)  $\mu_{\mathbf{d}, \mathbf{r}}(I) = 1$ ,
- (ii)  $\mu_{\mathbf{d}, \mathbf{r}}(I_1(n)) = d_n, \quad 0 \leq n \leq q-1$ ,
- (iii) for  $k \geq 2$ ,

$$\mu_{\mathbf{d}, \mathbf{r}}(I_k(n)) = \mu_{\mathbf{d}, \mathbf{r}}(I_{k-1}(j)) \times r_{\sigma^j(l)}, \quad 0 \leq n \leq q^k - 1,$$

where  $j$  and  $l$  are integers with  $n = qj + l$  ( $0 \leq l \leq q-1$ ). The distribution function  $L_{\mathbf{d}, \mathbf{r}}$  of  $\mu_{\mathbf{d}, \mathbf{r}}$  is defined by

$$L_{\mathbf{d}, \mathbf{r}}(x) = \mu_{\mathbf{d}, \mathbf{r}}([0, x]), \quad x \in I.$$

For simplicity, we use the abbreviation  $L_{\mathbf{r}}$  for  $L_{\mathbf{d}, \mathbf{r}}$ .

The measure  $\mu_{\mathbf{d}, \mathbf{r}}$  is a generalization of the multinomial measure (see [4]) and the Gray measure (see [2]).

There is an interesting relation between  $L_{\mathbf{r}}(x)$  and the exponential sum for a certain coding system related to paperfolding sequences (see [3, Theorem 1]). Moreover, since  $L_{\mathbf{r}}(x)$  is an analytic function of  $\mathbf{r}$  (see [3, Theorem 2]), the power sums for the coding system are related to the higher order derivatives of  $L_{\mathbf{r}}(x)$  with respect to  $\mathbf{r}$  (see [3, Theorem 3]).

[3, Theorem 4] states that the higher order derivatives of  $L_{\mathbf{r}}(x)$  with respect to  $\mathbf{r}$  are expressed by a generalization of the Takagi function. To describe [3, Theorem 4], we prepare several notations. Let  $\mathbf{q}$ ,  $\mathbf{e}_l$ , and  $\mathbf{u}$  be vectors with

$$\mathbf{q} = \underbrace{(1/q, \dots, 1/q)}_{q-1},$$

$$\mathbf{e}_l = \underbrace{(0, \dots, 0, \overset{l}{1}, 0, \dots, 0)}_{q-1}, \quad 0 \leq l \leq q-2,$$

$$\mathbf{u} = (u_0, \dots, u_{q-2}), \quad u_l \in \mathbf{N} \cup \{0\},$$

and define

$$|\mathbf{u}| = u_0 + u_1 + \cdots + u_{q-2}, \quad \mathbf{u}! = \prod_{l=0}^{q-2} u_l!.$$

For  $n \in \mathbf{N} \cup \{0\}$ , let  $\mathbf{r}_{\sigma^n}$  be the vector with

$$\mathbf{r}_{\sigma^n} = (r_{\sigma^n(0)}, \dots, r_{\sigma^n(q-2)}).$$

For a set  $S$ , let  $\mathbf{1}_S$  be the indicator function of  $S$ . Define the function  $\Phi_l$  on  $I$  by

$$\Phi_l = \sum_{j=0}^{q-1} \mathbf{1}_{I_2(qj+\sigma^{-j}(l))}, \quad 0 \leq l \leq q-1.$$

Let  $\phi(x)$  be the function on  $I$  such that  $\phi(x) = qx \pmod{1}$  with  $0 \leq \phi(x) < 1$  for  $x \in [0, 1)$  and  $\phi(1) = 1$ . We use the notation

$$f \circ \phi^j(x) = f(\underbrace{\phi(\phi(\cdots \phi(x)))}_j)$$

for any function  $f$ . We denote the Lebesgue measure on  $I$  by  $\mu$ .

**Definition 2.** The generalized Takagi function  $\mathcal{T}_{\mathbf{d}, \mathbf{r}, \mathbf{u}}(x)$  is defined as follows.

(i) If  $\mathbf{u} = \mathbf{e}_l$ , then

$$\mathcal{T}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l}(x) = \frac{1}{q} \sum_{j=0}^{\infty} \sum_{n=0}^{q^j-1} \mu_{\mathbf{d}, \mathbf{r}}(I_j(n)) \mathbf{1}_{I_j(n)}(x) \int_0^{\phi^j(x)} \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) d\mu_{\mathbf{r}_{\sigma^n}, \mathbf{r}}.$$

(ii) If  $|\mathbf{u}| \geq 2$ , then

$$\begin{aligned} \mathcal{T}_{\mathbf{d}, \mathbf{r}, \mathbf{u}}(x) &= \sum_{j=0}^{\infty} \sum_{\substack{\alpha=0 \\ u_\alpha > 0}}^{q-2} \left( \left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) \right) \\ &\quad \times \sum_{n=0}^{q^{j+1}-1} \mu_{\mathbf{d}, \mathbf{r}}(I_{j+1}(n)) \mathbf{1}_{I_{j+1}(n)}(x) \left( \mathcal{T}_{\mathbf{r}_{\sigma^n}, \mathbf{r}, \mathbf{u} - \mathbf{e}_\alpha} \circ \phi^{j+1}(x) \right). \end{aligned}$$

Then the higher order derivatives of  $L_{\mathbf{r}}(x)$  with respect to  $\mathbf{r}$  are expressed as the following.

**Theorem 1.** ([3, Theorem 4]) (i) If  $\mathbf{u} = \mathbf{e}_l$ , then

$$\frac{1}{q} \frac{\partial}{\partial r_l} L_{\mathbf{r}}(x) = (\mathbf{1}_{I_1(l)}(x) - \mathbf{1}_{I_1(q-1)}(x))(L_{\mathbf{q}, \mathbf{r}}(x) - x)$$

$$+ \left( \sum_{n=0}^{q-1} r_n \mathbf{1}_{I_1(n)}(x) \right) q \mathcal{T}_{\mathbf{q}, \mathbf{r}, \mathbf{e}_l}(x) + \int_0^x (\mathbf{1}_{I_1(l)} - \mathbf{1}_{I_1(q-1)}) d\mu.$$

(ii) If  $|\mathbf{u}| \geq 2$ , then

$$\begin{aligned} \frac{1}{q\mathbf{u}!} \frac{\partial^{u_0+\dots+u_{q-2}}}{\partial r_0^{u_0} \dots \partial r_{q-2}^{u_{q-2}}} L_{\mathbf{r}}(x) &= \sum_{\substack{j=0 \\ u_j > 0}}^{q-2} (\mathbf{1}_{I_1(j)}(x) - \mathbf{1}_{I_1(q-1)}(x)) q \mathcal{T}_{\mathbf{q}, \mathbf{r}, \mathbf{u}-\mathbf{e}_j}(x) \\ &\quad + \left( \sum_{n=0}^{q-1} r_n \mathbf{1}_{I_1(n)}(x) \right) q \mathcal{T}_{\mathbf{q}, \mathbf{r}, \mathbf{u}}(x). \end{aligned}$$

In [3], we only gave the sketch of the proof of the above Theorem 1, because the complete proof is very long. The purpose of this paper is to give the complete proof of Theorem 1.

Finally, we mention the previous works on studying relations between higher order derivatives of distribution functions and Takagi functions. Hata–Yamaguti [1] is the first work clarifying a relation between the first order derivative of  $L_{\mathbf{r}}(x)$  with respect to  $\mathbf{r}$  and the usual Takagi function in the dyadic case. In [5], Hata–Yamaguti’s result is studied from a viewpoint of the binomial measure, and, in [4], it is generalized in the  $q$ -adic case, in which the multinomial measure and its distribution function play essential roles. In Kobayashi [2], the Gray measure and its distribution function are studied from a viewpoint of [5] and [4]. Since the measure  $\mu_{\mathbf{d}, \mathbf{r}}$  is a generalization of the multinomial measure and Gray measure, Theorem 1 is a natural generalization of the results obtained in [1], [5], [4], and [2].

## 2 Preliminary lemmas

For a fixed  $k \in \mathbf{N}$ , any integer  $n$  with  $0 \leq n \leq q^k - 1$  is expressed as  $n = \sum_{i=0}^{k-1} n_i q^i$ , where  $n_i \in \{0, 1, \dots, q-1\}$ . We use the abbreviation  $n = n_{k-1} \dots n_0$  for  $n = \sum_{i=0}^{k-1} n_i q^i$ , in which the length of the word is always equal to  $k$ , and identify  $I_k(n)$  with  $I_k(n_{k-1} \dots n_0)$ .

Firstly, we study a relation between  $\phi^i$  and  $\mu_{\mathbf{d}, \mathbf{r}}$ . We note a simple fact

$$I_{i+k}(b_{i-1} \dots b_0 a_{k-1} \dots a_0) \subset I_i(c_{i-1} \dots c_0) \Leftrightarrow b_{i-1} \dots b_0 = c_{i-1} \dots c_0.$$

**Lemma 1.** *We have*

$$\phi^i \left( \bigcup_{0 \leq b_0, \dots, b_{i-1} \leq q-1} I_{i+k}(b_{i-1} \dots b_0 a_{k-1} \dots a_0) \right) = I_k(a_{k-1} \dots a_0).$$

*Proof.* By the definition of  $\phi$ , we have

$$\phi \left( \bigcup_{0 \leq b_0, \dots, b_{i-1} \leq q-1} I_{i+k}(b_{i-1} \dots b_0 a_{k-1} \dots a_0) \right)$$

$$= \bigcup_{0 \leq b_0, \dots, b_{i-2} \leq q-1} I_{i-1+k}(b_{i-2} \cdots b_0 a_{k-1} \cdots a_0).$$

Repeating this  $i$  times, we obtain the assertion.  $\square$

Lemma 1 is equivalent to the following.

**Lemma 2.** *We have*

$$\mathbf{1}_{I_k(a_{k-1} \cdots a_0)} \circ \phi^i = \mathbf{1}_{\bigcup_{0 \leq b_0, \dots, b_{i-1} \leq q-1} I_{i+k}(b_{i-1} \cdots b_0 a_{k-1} \cdots a_0)}.$$

**Lemma 3.** *We have*

$$\mu_{\mathbf{d}, \mathbf{r}}(I_{i+k}(b_{i-1} \cdots b_0 a_{k-1} \cdots a_0)) = \mu_{\mathbf{d}, \mathbf{r}}(I_i(b_{i-1} \cdots b_0)) \mu_{\mathbf{r}_{\sigma^{b_0}}, \mathbf{r}}(I_k(a_{k-1} \cdots a_0)).$$

*Proof.* It follows from Definition 1 and the assumption  $\sigma^q = \text{id}$  that

$$\mu_{\mathbf{d}, \mathbf{r}}(I_{i+k}(b_{i-1} \cdots b_0 a_{k-1} \cdots a_0)) = \mu_{\mathbf{d}, \mathbf{r}}(I_i(b_{i-1} \cdots b_0)) r_{\sigma^{b_0}(a_{k-1})} r_{\sigma^{a_{k-1}}(a_{k-2})} \cdots r_{\sigma^{a_1}(a_0)}$$

and

$$\mu_{\mathbf{r}_{\sigma^{b_0}}, \mathbf{r}}(I_k(a_{k-1} \cdots a_0)) = r_{\sigma^{b_0}(a_{k-1})} r_{\sigma^{a_{k-1}}(a_{k-2})} \cdots r_{\sigma^{a_1}(a_0)}.$$

Hence we obtain the assertion.  $\square$

**Lemma 4.** *For any  $i \in \mathbf{N}$ ,  $a \in \mathbf{N} \cup \{0\}$  with  $0 \leq a \leq q^i - 1$ , and  $x \in I_i(a_{i-1} \cdots a_0)$ , we have*

$$\mathbf{1}_{[0, \phi^i(x)]}(\phi^i(y)) \times \mathbf{1}_{I_i(a_{i-1} \cdots a_0)}(y) = \mathbf{1}_{[0, x]}(y) \times \mathbf{1}_{I_i(a_{i-1} \cdots a_0)}(y). \quad (1)$$

*Proof.* We prove this by induction on  $i$ . When  $i = 1$ , we have for  $x \in I_1(a_0)$

$$\phi(y) \in [0, \phi(x)] \Leftrightarrow y \in \bigcup_{m=0}^{q-1} \left[ \frac{m}{q}, \frac{m}{q} + \left| x - \frac{a_0}{q} \right| \right],$$

and hence

$$\phi(y) \in [0, \phi(x)] \text{ and } y \in I_1(a_0) \Leftrightarrow y \in [0, x] \cap I_1(a_0),$$

from which we get

$$\mathbf{1}_{[0, \phi(x)]}(\phi(y)) \times \mathbf{1}_{I_1(a_0)}(y) = \mathbf{1}_{[0, x]}(y) \times \mathbf{1}_{I_1(a_0)}(y). \quad (2)$$

By Lemma 2, we have

$$\mathbf{1}_{I_1(a_0)}(\phi^i(y)) \times \mathbf{1}_{I_i(a_i \cdots a_1)}(y) = \mathbf{1}_{I_{i+1}(a_i \cdots a_1 a_0)}(y). \quad (3)$$

Therefore, if  $x \in I_{i+1}(a_i \cdots a_1 a_0)$  and (1) holds for  $i$ , by (3), (2), and Lemma 2, we have

$$\begin{aligned}
& \mathbf{1}_{[0, \phi^{i+1}(x)]}(\phi^{i+1}(y)) \times \mathbf{1}_{I_{i+1}(a_i \cdots a_1 a_0)}(y) \\
&= \mathbf{1}_{[0, \phi(\phi^i(x))]}(\phi(\phi^i(y))) \times \mathbf{1}_{I_1(a_0)}(\phi^i(y)) \times \mathbf{1}_{I_i(a_i \cdots a_1)}(y) \\
&= \mathbf{1}_{[0, \phi^i(x)]}(\phi^i(y)) \times \mathbf{1}_{I_1(a_0)}(\phi^i(y)) \times \mathbf{1}_{I_i(a_i \cdots a_1)}(y) \\
&= \mathbf{1}_{[0, x]}(y) \times \mathbf{1}_{I_i(a_i \cdots a_1)}(y) \times \mathbf{1}_{I_1(a_0)}(\phi^i(y)) \\
&= \mathbf{1}_{[0, x]}(y) \times \mathbf{1}_{I_{i+1}(a_i \cdots a_1 a_0)}(y).
\end{aligned}$$

This completes the proof.  $\square$

For any bounded  $\mathcal{F}$ -measurable function  $f$ , let

$$\begin{aligned}
E_{\mu_{\mathbf{d}, \mathbf{r}}}(f) &= \int_I f d\mu_{\mathbf{d}, \mathbf{r}}, \\
E_{\mu_{\mathbf{d}, \mathbf{r}}}(f; I_k(n)) &= \int_{I_k(n)} f d\mu_{\mathbf{d}, \mathbf{r}}.
\end{aligned}$$

Lemmas 5 and 6 show that a kind of integration by substitution is valid.

**Lemma 5.** *For any  $i \in \mathbf{N}$ ,  $a \in \mathbf{N} \cup \{0\}$  with  $0 \leq a \leq q^i - 1$ , and a bounded  $\mathcal{F}$ -measurable function  $f$ , we have*

$$E_{\mu_{\mathbf{d}, \mathbf{r}}}(f \circ \phi^i; I_i(a_{i-1} \cdots a_0)) = \mu_{\mathbf{d}, \mathbf{r}}(I_i(a_{i-1} \cdots a_0)) E_{\mu_{\mathbf{r}_{\sigma^{a_0}}, \mathbf{r}}}(f).$$

*Proof.* Since a bounded  $\mathcal{F}$ -measurable function can be approximated by step functions, it suffices to show the equality for  $f = \mathbf{1}_{I_j(c_{j-1} \cdots c_0)}$ . By Lemmas 2 and 3, we have

$$\begin{aligned}
& E_{\mu_{\mathbf{d}, \mathbf{r}}}(\mathbf{1}_{I_j(c_{j-1} \cdots c_0)} \circ \phi^i; I_i(a_{i-1} \cdots a_0)) \\
&= \int_I \mathbf{1}_{\bigcup_{0 \leq b_0, \dots, b_{i-1} \leq q-1} I_{i+j}(b_{i-1} \cdots b_0 c_{j-1} \cdots c_0)} \times \mathbf{1}_{I_i(a_{i-1} \cdots a_0)} d\mu_{\mathbf{d}, \mathbf{r}} \\
&= \int_I \mathbf{1}_{I_{i+j}(a_{i-1} \cdots a_0 c_{j-1} \cdots c_0)} d\mu_{\mathbf{d}, \mathbf{r}} \\
&= \mu_{\mathbf{d}, \mathbf{r}}(I_{i+j}(a_{i-1} \cdots a_0 c_{j-1} \cdots c_0)) \\
&= \mu_{\mathbf{d}, \mathbf{r}}(I_i(a_{i-1} \cdots a_0)) E_{\mu_{\mathbf{r}_{\sigma^{a_0}}, \mathbf{r}}}(\mathbf{1}_{I_j(c_{j-1} \cdots c_0)}).
\end{aligned}$$

$\square$

**Lemma 6.** *For any  $i \in \mathbf{N}$ ,  $a \in \mathbf{N} \cup \{0\}$  with  $0 \leq a \leq q^i - 1$ , a bounded  $\mathcal{F}$ -measurable function  $f$ , and  $x \in I_i(a_{i-1} \cdots a_0)$ , we have*

$$E_{\mu_{\mathbf{d}, \mathbf{r}}}(f \circ \phi^i; I_i(a_{i-1} \cdots a_0) \cap [0, x]) = \mu_{\mathbf{d}, \mathbf{r}}(I_i(a_{i-1} \cdots a_0)) E_{\mu_{\mathbf{r}_{\sigma^{a_0}}, \mathbf{r}}}(f; [0, \phi^i(x)]).$$

*Proof.* By Lemmas 4 and 5, we obtain

$$\begin{aligned}
& \mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(f \circ \phi^i; I_i(a_{i-1} \cdots a_0) \cap [0, x]) \\
&= \mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}((f \circ \phi^i) \times \mathbf{1}_{[0, x]}; I_i(a_{i-1} \cdots a_0)) \\
&= \mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}((f \circ \phi^i) \times (\mathbf{1}_{[0, \phi^i(x)]} \circ \phi^i); I_i(a_{i-1} \cdots a_0)) \\
&= \mu_{\mathbf{d}, \mathbf{r}}(I_i(a_{i-1} \cdots a_0)) \mathbb{E}_{\mu_{\mathbf{r}_{\sigma^{a_0}}, \mathbf{r}}}(f \times \mathbf{1}_{[0, \phi^i(x)]}). \quad \square
\end{aligned}$$

Next, we discuss the conditional expectation  $\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(\cdot | \mathcal{F}_k)$ . For a bounded  $\mathcal{F}$ -measurable function  $g$ ,  $\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g | \mathcal{F}_k)$  is defined to be the  $\mathcal{F}_k$ -measurable function such that

$$\int_G \mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g | \mathcal{F}_k) d\mu_{\mathbf{d}, \mathbf{r}} = \int_G g d\mu_{\mathbf{d}, \mathbf{r}}, \quad \text{for all } G \in \mathcal{F}_k.$$

Since  $\mathcal{F}_k$  is the finite set and  $\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g | \mathcal{F}_k)$  is  $\mathcal{F}_k$ -measurable,  $\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g | \mathcal{F}_k)$  is a step function with constant values on  $I_k(n)$ 's. In fact, it is written explicitly as

$$\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g | \mathcal{F}_k) = \sum_{n=0}^{q^k-1} \frac{\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g; I_k(n))}{\mu_{\mathbf{d}, \mathbf{r}}(I_k(n))} \mathbf{1}_{I_k(n)}. \quad (4)$$

**Lemma 7.** *Let  $g$  be a bounded  $\mathcal{F}$ -measurable function. If  $h$  is a  $\mathcal{F}_k$ -measurable function, and  $g$  satisfies  $\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g | \mathcal{F}_k) = 0$ , then*

$$\int_0^x hg d\mu_{\mathbf{d}, \mathbf{r}} = h(x) \int_0^x g d\mu_{\mathbf{d}, \mathbf{r}}.$$

*Proof.* By (4),  $\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g | \mathcal{F}_k) = 0$  is equivalent to  $\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g; I_k(n)) = 0$  for every  $n$ . Since  $h$  is  $\mathcal{F}_k$ -measurable, it takes a constant value  $C_n$  on  $I_k(n)$ . Hence

$$\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(hg; I_k(n)) = C_n \mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g; I_k(n)) = 0.$$

Thus we obtain for  $x \in I_k(m)$

$$\begin{aligned}
\int_0^x hg d\mu_{\mathbf{d}, \mathbf{r}} &= \mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(hg; I_k(m) \cap [0, x]) \\
&= h(x) \mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}}(g; I_k(m) \cap [0, x]) = h(x) \int_0^x g d\mu_{\mathbf{d}, \mathbf{r}}.
\end{aligned}$$

Since the equality is independent of  $m$ , it is valid for  $x \in I$ .  $\square$

### 3 The Radon-Nikodym derivative on the finite set

Let  $\mathbf{e} = (e_0, \dots, e_{q-2})$  and  $\mathbf{s} = (s_0, \dots, s_{q-2})$  be vectors whose components satisfy the same conditions as those of  $\mathbf{d}$  in Definition 1, and set  $e_{q-1} = 1 - \sum_{j=0}^{q-2} e_j$  and  $s_{q-1} = 1 - \sum_{j=0}^{q-2} s_j$ .

**Definition 3.** The function  $Z \left[ \begin{smallmatrix} \mathbf{e} & \mathbf{s} \\ \mathbf{d} & \mathbf{r} \end{smallmatrix}; k \right] : I \rightarrow \mathbf{R}$  is defined by

$$Z \left[ \begin{smallmatrix} \mathbf{e} & \mathbf{s} \\ \mathbf{d} & \mathbf{r} \end{smallmatrix}; k \right] = \sum_{n=0}^{q^k-1} \frac{\mu_{\mathbf{e},\mathbf{s}}(I_k(n))}{\mu_{\mathbf{d},\mathbf{r}}(I_k(n))} \mathbf{1}_{I_k(n)}, \quad k \in \mathbf{N} \cup \{0\}.$$

**Remark 1.**  $Z \left[ \begin{smallmatrix} \mathbf{e} & \mathbf{s} \\ \mathbf{d} & \mathbf{r} \end{smallmatrix}; k \right]$  is the so-called Radon-Nikodym derivative  $d\mu_{\mathbf{e},\mathbf{s}}/d\mu_{\mathbf{d},\mathbf{r}}$  on  $\mathcal{F}_k$ .

We identify  $I_k(n)$  with  $I_k(n_{k-1} \cdots n_0)$  as in the previous section.

**Definition 4.** The function  $W \left[ \begin{smallmatrix} \mathbf{s} \\ \mathbf{r} \end{smallmatrix} \right] : I \rightarrow \mathbf{R}$  is defined by

$$W \left[ \begin{smallmatrix} \mathbf{s} \\ \mathbf{r} \end{smallmatrix} \right] = \sum_{0 \leq b_0, b_1 \leq q-1} \frac{s_{\sigma^{b_1}(b_0)}}{r_{\sigma^{b_1}(b_0)}} \mathbf{1}_{I_2(b_1 b_0)}.$$

The following propositions have been proved in [3].

**Proposition 1.** *We have*

$$L_{\mathbf{e},\mathbf{s}}(x) = \lim_{k \rightarrow \infty} \int_0^x Z \left[ \begin{smallmatrix} \mathbf{e} & \mathbf{s} \\ \mathbf{d} & \mathbf{r} \end{smallmatrix}; k \right] d\mu_{\mathbf{d},\mathbf{r}},$$

where the convergence is uniform for  $\mathbf{e} = (e_0, \dots, e_{q-2})$  and  $\mathbf{s} = (s_0, \dots, s_{q-2})$ .

**Proposition 2.** *For  $k \geq 1$ , we have*

$$Z \left[ \begin{smallmatrix} \mathbf{e} & \mathbf{s} \\ \mathbf{d} & \mathbf{r} \end{smallmatrix}; k+1 \right] = \left( \prod_{i=0}^{k-1} W \left[ \begin{smallmatrix} \mathbf{s} \\ \mathbf{r} \end{smallmatrix} \right] \circ \phi^i \right) Z \left[ \begin{smallmatrix} \mathbf{e} & \mathbf{s} \\ \mathbf{d} & \mathbf{r} \end{smallmatrix}; 1 \right].$$

### 4 Higher order derivatives of distribution functions

Firstly, we study a relation between  $L_{\mathbf{s}}(x)(= L_{\mathbf{s},\mathbf{s}}(x))$  and  $L_{\mathbf{q},\mathbf{s}}(x)$ .

**Lemma 8.** *We have*

$$L_{\mathbf{s}}(x) = \left( q \sum_{n=0}^{q-1} s_n \mathbf{1}_{I_1(n)}(x) \right) (L_{\mathbf{q},\mathbf{s}}(x) - x) + q \int_0^x \sum_{n=0}^{q-1} s_n \mathbf{1}_{I_1(n)} d\mu,$$

where  $\mu$  is the Lebesgue measure on  $I$ .



*Proof.* Let  $x \in I_1(m)$  ( $0 \leq m \leq q-1$ ). Then it follows that

$$\begin{aligned}
L_{\mathbf{s}}(x) &= L_{\mathbf{s}}\left(\frac{m}{q}\right) + \mu_{\mathbf{s},\mathbf{s}}\left(\left(\frac{m}{q}, x\right]\right) \\
&= \sum_{n=0}^{m-1} s_n + \frac{s_m}{1/q} \mu_{\mathbf{q},\mathbf{s}}\left(\left(\frac{m}{q}, x\right]\right) \\
&= qs_m(L_{\mathbf{q},\mathbf{s}}(x) - x) + \sum_{n=0}^{m-1} s_n + qs_m\left(x - \frac{m}{q}\right).
\end{aligned} \tag{5}$$

Noting that, for  $x \in I_1(m)$ ,

$$\int_0^x \mathbf{1}_{I_1(n)} d\mu = \begin{cases} 0, & n > m, \\ x - \frac{n}{q}, & n = m, \\ \frac{1}{q}, & n < m, \end{cases}$$

we have

$$\sum_{n=0}^{m-1} s_n + qs_m\left(x - \frac{m}{q}\right) = \sum_{n=0}^{q-1} qs_n \int_0^x \mathbf{1}_{I_1(n)} d\mu. \tag{6}$$

Substituting (6) into (5) and replacing the range of the variable  $x$  to  $I$ , we obtain the assertion.  $\square$

By Lemma 8, we have easily the following relation between the higher order derivative of  $L_{\mathbf{s}}(x)$  and that of  $L_{\mathbf{q},\mathbf{s}}(x)$ .

**Lemma 9.** (i) If  $\mathbf{u} = \mathbf{e}_l$ , then

$$\begin{aligned}
\frac{1}{q} \frac{\partial}{\partial s_l} L_{\mathbf{s}}(x) &= (\mathbf{1}_{I_1(l)}(x) - \mathbf{1}_{I_1(q-1)}(x))(L_{\mathbf{q},\mathbf{s}}(x) - x) \\
&\quad + \left( \sum_{n=0}^{q-1} s_n \mathbf{1}_{I_1(n)}(x) \right) \frac{\partial}{\partial s_l} L_{\mathbf{q},\mathbf{s}}(x) + \int_0^x (\mathbf{1}_{I_1(l)} - \mathbf{1}_{I_1(q-1)}) d\mu.
\end{aligned}$$

(ii) If  $|\mathbf{u}| \geq 2$ , then

$$\begin{aligned}
\frac{1}{q} \frac{\partial^{u_0+\dots+u_{q-2}}}{\partial s_0^{u_0} \dots \partial s_{q-2}^{u_{q-2}}} L_{\mathbf{s}}(x) &= \sum_{\substack{j=0 \\ u_j > 0}}^{q-2} u_j (\mathbf{1}_{I_1(j)}(x) - \mathbf{1}_{I_1(q-1)}(x)) \\
&\quad \times \frac{\partial^{u_0+\dots+u_{j-1}+(u_j-1)+u_{j+1}+\dots+u_{q-2}}}{\partial s_0^{u_0} \dots \partial s_{j-1}^{u_{j-1}} \partial s_j^{u_j-1} \partial s_{j+1}^{u_{j+1}} \dots \partial s_{q-2}^{u_{q-2}}} L_{\mathbf{q},\mathbf{s}}(x) \\
&\quad + \left( \sum_{n=0}^{q-1} s_n \mathbf{1}_{I_1(n)}(x) \right) \frac{\partial^{u_0+\dots+u_{q-2}}}{\partial s_0^{u_0} \dots \partial s_{q-2}^{u_{q-2}}} L_{\mathbf{q},\mathbf{s}}(x).
\end{aligned}$$

Next, we study the higher order derivative of  $L_{\mathbf{q},\mathbf{s}}(x)$ . Let  $\psi_{\mathbf{u}} : \{1, 2, \dots, |\mathbf{u}|\} \rightarrow \{0, 1, \dots, q-2\}$  be a mapping such that  $\#\{m; \psi_{\mathbf{u}}(m) = j\} = u_j$ , which is the same one as that of [4]. For example, if  $q = 4$ ,  $\mathbf{u} = (u_0, u_1, u_2) = (1, 2, 0)$ , then  $\psi_{\mathbf{u}} : \{1, 2, 3\} \rightarrow \{0, 1, 2\}$  is a mapping satisfying  $\#\{m; \psi_{\mathbf{u}}(m) = 0\} = 1$ ,  $\#\{m; \psi_{\mathbf{u}}(m) = 1\} = 2$ , and  $\#\{m; \psi_{\mathbf{u}}(m) = 2\} = 0$ . In fact,  $\psi_{\mathbf{u}}$  is one of three mappings

$$\begin{cases} \psi_{\mathbf{u}}(1) = 0, \\ \psi_{\mathbf{u}}(2) = 1, \\ \psi_{\mathbf{u}}(3) = 1, \end{cases} \quad \begin{cases} \psi_{\mathbf{u}}(1) = 1, \\ \psi_{\mathbf{u}}(2) = 0, \\ \psi_{\mathbf{u}}(3) = 1, \end{cases} \quad \begin{cases} \psi_{\mathbf{u}}(1) = 1, \\ \psi_{\mathbf{u}}(2) = 1, \\ \psi_{\mathbf{u}}(3) = 0. \end{cases}$$

**Lemma 10.** *We have*

$$\begin{aligned} & \frac{\partial^{u_0+\dots+u_{q-2}}}{\partial s_0^{u_0} \dots \partial s_{q-2}^{u_{q-2}}} L_{\mathbf{q},\mathbf{s}}(x) \Big|_{\mathbf{s}=\mathbf{r}} \\ &= \mathbf{u}! \lim_{k \rightarrow \infty} \sum_{0 \leq i_1 < \dots < i_{|\mathbf{u}|} \leq k-2} \sum_{\psi_{\mathbf{u}}} \int_0^x \prod_{m=1}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi_{\mathbf{u}}(m)}}{r_{\psi_{\mathbf{u}}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i_m} d\mu_{\mathbf{q},\mathbf{r}}, \end{aligned}$$

where the sum  $\sum_{\psi_{\mathbf{u}}}$  is taken over all  $\psi_{\mathbf{u}}$ 's.

*Proof.* By Propositions 1 and 2 with  $\mathbf{e} = \mathbf{d} = \mathbf{q}$ , we have

$$L_{\mathbf{q},\mathbf{s}}(x) = \lim_{k \rightarrow \infty} \int_0^x \prod_{i=0}^{k-2} W \begin{bmatrix} \mathbf{s} \\ \mathbf{r} \end{bmatrix} \circ \phi^i d\mu_{\mathbf{q},\mathbf{r}}. \quad (7)$$

From the definitions of  $W \begin{bmatrix} \mathbf{s} \\ \mathbf{r} \end{bmatrix}$  and  $\Phi_l$ , it follows that

$$W \begin{bmatrix} \mathbf{s} \\ \mathbf{r} \end{bmatrix} \circ \phi^i = \sum_{l=0}^{q-1} \frac{s_l}{r_l} (\Phi_l \circ \phi^i). \quad (8)$$

Let  $\mathcal{K}_{\mathbf{s},i} = \sum_{l=0}^{q-1} s_l (\Phi_l \circ \phi^i)$ . We show the equality

$$\sum_{l=0}^{q-1} \frac{s_l}{r_l} (\Phi_l \circ \phi^i) = \frac{\mathcal{K}_{\mathbf{s},i}}{\mathcal{K}_{\mathbf{r},i}}. \quad (9)$$

Since  $\sum_{l=0}^{q-1} \Phi_l = 1$ , it holds that  $\sum_{l=0}^{q-1} \Phi_l \circ \phi^i = 1$ . For any  $x \in I$ , there exists a unique  $m$  such that  $\Phi_m \circ \phi^i(x) = 1$ ,  $\Phi_l \circ \phi^i(x) = 0$  ( $l \neq m$ ), and hence, both of  $\sum_{l=0}^{q-1} \frac{s_l}{r_l} (\Phi_l \circ \phi^i(x))$  and  $\frac{\mathcal{K}_{\mathbf{s},i}}{\mathcal{K}_{\mathbf{r},i}}(x)$  are  $\frac{s_m}{r_m}$ . Combining (7), (8), and (9), we have

$$L_{\mathbf{q},\mathbf{s}}(x) = \lim_{k \rightarrow \infty} \int_0^x \prod_{i=0}^{k-2} \frac{\mathcal{K}_{\mathbf{s},i}}{\mathcal{K}_{\mathbf{r},i}} d\mu_{\mathbf{q},\mathbf{r}}. \quad (10)$$

For  $a$  with  $0 \leq a \leq q-2$ ,

$$\frac{\partial}{\partial s_a} \frac{\mathcal{K}_{s,i}}{\mathcal{K}_{r,i}} = \frac{(\Phi_a - \Phi_{q-1}) \circ \phi^i}{\mathcal{K}_{r,i}}.$$

By the same argument as in [4, pp.459–460],

$$\begin{aligned} & \frac{\partial^{u_0+\dots+u_{q-2}}}{\partial s_0^{u_0} \dots \partial s_{q-2}^{u_{q-2}}} \int_0^x \prod_{i=0}^{k-2} \frac{\mathcal{K}_{s,i}}{\mathcal{K}_{r,i}} d\mu_{q,r} \\ &= \mathbf{u}! \sum_{0 \leq i_1 < \dots < i_{|\mathbf{u}|} \leq k-2} \sum_{\psi_{\mathbf{u}}} \int_0^x \left( \prod_{m=1}^{|\mathbf{u}|} \frac{(\Phi_{\psi_{\mathbf{u}}(m)} - \Phi_{q-1}) \circ \phi^{i_m}}{\mathcal{K}_{s,i_m}} \right) \times \left( \prod_{i=0}^{k-2} \frac{\mathcal{K}_{s,i}}{\mathcal{K}_{r,i}} \right) d\mu_{q,r}. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{\partial^{u_0+\dots+u_{q-2}}}{\partial s_0^{u_0} \dots \partial s_{q-2}^{u_{q-2}}} \int_0^x \prod_{i=0}^{k-2} \frac{\mathcal{K}_{s,i}}{\mathcal{K}_{r,i}} d\mu_{q,r} \Big|_{s=r} \\ &= \mathbf{u}! \sum_{0 \leq i_1 < \dots < i_{|\mathbf{u}|} \leq k-2} \sum_{\psi_{\mathbf{u}}} \int_0^x \prod_{m=1}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi_{\mathbf{u}}(m)}}{r_{\psi_{\mathbf{u}}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i_m} d\mu_{q,r}. \end{aligned} \quad (11)$$

From (10) and (11), the assertion follows.  $\square$

By Lemmas 9 and 10, and  $u_j(\mathbf{u} - \mathbf{e}_j)! = \mathbf{u}!$ , we obtain the following.

**Proposition 3.** (i) *If  $\mathbf{u} = \mathbf{e}_l$ , then*

$$\begin{aligned} \frac{1}{q} \frac{\partial}{\partial s_l} L_s(x) \Big|_{s=r} &= (\mathbf{1}_{I_1(l)}(x) - \mathbf{1}_{I_1(q-1)}(x))(L_{q,r}(x) - x) \\ &+ \left( \sum_{n=0}^{q-1} r_n \mathbf{1}_{I_1(n)}(x) \right) \lim_{k \rightarrow \infty} \sum_{0 \leq j \leq k-2} \int_0^x \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j d\mu_{q,r} \\ &+ \int_0^x (\mathbf{1}_{I_1(l)} - \mathbf{1}_{I_1(q-1)}) d\mu. \end{aligned}$$

(ii) *If  $|\mathbf{u}| \geq 2$ , then*

$$\begin{aligned} \frac{1}{q\mathbf{u}!} \frac{\partial^{u_0+\dots+u_{q-2}}}{\partial s_0^{u_0} \dots \partial s_{q-2}^{u_{q-2}}} L_s(x) \Big|_{s=r} &= \sum_{\substack{j=0 \\ u_j > 0}}^{q-2} (\mathbf{1}_{I_1(j)}(x) - \mathbf{1}_{I_1(q-1)}(x)) \\ &\times \lim_{k \rightarrow \infty} \sum_{0 \leq i_1 < \dots < i_{|\mathbf{u}|-1} \leq k-2} \sum_{\psi_{\mathbf{u}-\mathbf{e}_j}} \int_0^x \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi_{\mathbf{u}-\mathbf{e}_j}(m)}}{r_{\psi_{\mathbf{u}-\mathbf{e}_j}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i_m} d\mu_{q,r} \\ &+ \left( \sum_{n=0}^{q-1} r_n \mathbf{1}_{I_1(n)}(x) \right) \lim_{k \rightarrow \infty} \sum_{0 \leq i_1 < \dots < i_{|\mathbf{u}|} \leq k-2} \sum_{\psi_{\mathbf{u}}} \int_0^x \prod_{m=1}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi_{\mathbf{u}}(m)}}{r_{\psi_{\mathbf{u}}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i_m} d\mu_{q,r}. \end{aligned}$$

## 5 A recursive relation for $\mathcal{D}_{\mathbf{d},\mathbf{r},\mathbf{u},k}(x)$

Based on the expression of Proposition 3, we introduce the function  $\mathcal{D}_{\mathbf{d},\mathbf{r},\mathbf{u},k}$ .

**Definition 5.** The function  $\mathcal{D}_{\mathbf{d},\mathbf{r},\mathbf{u},k} : I \rightarrow \mathbf{R}$  is defined by

$$\mathcal{D}_{\mathbf{d},\mathbf{r},\mathbf{u},k}(x) = \frac{1}{q} \sum_{0 \leq i_1 < \dots < i_{|\mathbf{u}|} \leq k} \sum_{\psi_{\mathbf{u}}} \int_0^x \prod_{m=1}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi_{\mathbf{u}}(m)}}{r_{\psi_{\mathbf{u}}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i_m} d\mu_{\mathbf{d},\mathbf{r}}.$$

We will give a recursive relation for  $\mathcal{D}_{\mathbf{d},\mathbf{r},\mathbf{u},k}(x)$  (see Proposition 4 below), which gives the definition of generalized Takagi functions.

**Lemma 11.** For any  $k, \beta \in \mathbf{N}$  with  $\beta + 2 > k$ , and integers  $l, k$  with  $0 \leq l \leq q-1$ ,  $0 \leq n \leq q^k - 1$ , we have

$$\mathbf{E}_{\mu_{\mathbf{d},\mathbf{r}}} \left( \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^\beta; I_k(n) \right) = 0.$$

*Proof.* For the  $q$ -adic representations  $n = n_{k-1} \dots n_0$  and  $qj + \sigma^{-j}(l) = j\sigma^{-j}(l)$ , we have

$$\begin{aligned} & \mathbf{E}_{\mu_{\mathbf{d},\mathbf{r}}} \left( \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^\beta; I_k(n_{k-1} \dots n_0) \right) \\ &= \frac{1}{r_l} \sum_{j=0}^{q-1} \mathbf{E}_{\mu_{\mathbf{d},\mathbf{r}}} (\mathbf{1}_{I_2(j\sigma^{-j}(l))} \circ \phi^\beta; I_k(n_{k-1} \dots n_0)) \\ & \quad - \frac{1}{r_{q-1}} \sum_{j=0}^{q-1} \mathbf{E}_{\mu_{\mathbf{d},\mathbf{r}}} (\mathbf{1}_{I_2(j\sigma^{-j}(q-1))} \circ \phi^\beta; I_k(n_{k-1} \dots n_0)). \end{aligned} \quad (12)$$

By Lemma 2,

$$\begin{aligned} & \sum_{j=0}^{q-1} \mathbf{E}_{\mu_{\mathbf{d},\mathbf{r}}} (\mathbf{1}_{I_2(j\sigma^{-j}(l))} \circ \phi^\beta; I_k(n_{k-1} \dots n_0)) \\ &= \mathbf{E}_{\mu_{\mathbf{d},\mathbf{r}}} (\mathbf{1}_{\bigcup_{0 \leq b_2, \dots, b_{\beta+1} \leq q-1} \bigcup_{b_1=0}^{q-1} I_{\beta+2}(b_{\beta+1} \dots b_2 b_1 \sigma^{-b_1}(l))}; I_k(n_{k-1} \dots n_0)) \\ &= \mathbf{E}_{\mu_{\mathbf{d},\mathbf{r}}} (\mathbf{1}_{\bigcup_{0 \leq b_1, \dots, b_{\beta-k+1} \leq q-1} I_{\beta+2}(n_{k-1} \dots n_0 b_{\beta-k+1} \dots b_1 \sigma^{-b_1}(l))}) \\ &= \sum_{0 \leq b_1, \dots, b_{\beta-k+1} \leq q-1} \mu_{\mathbf{d},\mathbf{r}}(I_{\beta+2}(n_{k-1} \dots n_0 b_{\beta-k+1} \dots b_1 \sigma^{-b_1}(l))). \end{aligned}$$

Here, by Lemma 3,

$$\begin{aligned} & \mu_{\mathbf{d},\mathbf{r}}(I_{\beta+2}(n_{k-1} \dots n_0 b_{\beta-k+1} \dots b_1 \sigma^{-b_1}(l))) \\ &= \mu_{\mathbf{d},\mathbf{r}}(I_{\beta+1}(n_{k-1} \dots n_0 b_{\beta-k+1} \dots b_1)) \mu_{\mathbf{r}_{\sigma^{b_1}},\mathbf{r}}(I_1(\sigma^{-b_1}(l))) \end{aligned}$$

$$= r_l \mu_{\mathbf{d}, \mathbf{r}}(I_{\beta+1}(n_{k-1} \cdots n_0 b_{\beta-k+1} \cdots b_1)),$$

and hence

$$\sum_{j=0}^{q-1} E_{\mu_{\mathbf{d}, \mathbf{r}}}(\mathbf{1}_{I_2(j\sigma^{-j}(l))} \circ \phi^\beta; I_k(n_{k-1} \cdots n_0)) = r_l \mu_{\mathbf{d}, \mathbf{r}}(I_k(n_{k-1} \cdots n_0)). \quad (13)$$

Substituting (13) into (12), we obtain the assertion.  $\square$

**Lemma 12.** *For any  $\mathbf{u}$  with  $|\mathbf{u}| \geq 2$  and  $k \in \mathbf{N}$ , let  $\{\beta_m\}_{m=1}^{|\mathbf{u}|}$  be a strictly increasing sequence with  $\beta_1 + 2 > k$ . Then we have*

$$E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi_{\mathbf{u}}(m)}}{r_{\psi_{\mathbf{u}}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_m}; I_k(n) \right) = 0$$

for every  $0 \leq n \leq q^k - 1$ .

*Proof.* Set  $\alpha = \psi_{\mathbf{u}}(1)$ . Then  $u_\alpha > 0$  by the definition of  $\psi_{\mathbf{u}}$ . We classify the set of  $\psi_{\mathbf{u}}$ 's by  $\alpha$ . By the definition of  $\psi_{\mathbf{u}}$ ,

$$\begin{aligned} \psi_{\mathbf{u}} : \{1, 2, \dots, |\mathbf{u}|\} &\longrightarrow \{0, 1, \dots, q-2\}, \\ \#\{2 \leq m \leq |\mathbf{u}|; \psi_{\mathbf{u}}(m) = j\} &= \begin{cases} u_j - 1, & \text{if } j = \alpha, \\ u_j, & \text{if } j \neq \alpha, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \psi_{\mathbf{u}-e_\alpha} : \{1, 2, \dots, |\mathbf{u}| - 1\} &\longrightarrow \{0, 1, \dots, q-2\}, \\ \#\{1 \leq m \leq |\mathbf{u}| - 1; \psi_{\mathbf{u}-e_\alpha}(m) = j\} &= \begin{cases} u_j - 1, & \text{if } j = \alpha, \\ u_j, & \text{if } j \neq \alpha. \end{cases} \end{aligned}$$

Hence, for any  $\psi_{\mathbf{u}}$  there exists a unique  $\psi_{\mathbf{u}-e_\alpha}$  such that

$$\psi_{\mathbf{u}}(m) = \psi_{\mathbf{u}-e_\alpha}(m-1), \quad 2 \leq m \leq |\mathbf{u}|. \quad (14)$$

It follows from (14) that

$$\begin{aligned} &E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi_{\mathbf{u}}(m)}}{r_{\psi_{\mathbf{u}}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_m}; I_k(n) \right) \\ &= E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \left( \left( \frac{\Phi_{\psi_{\mathbf{u}}(1)}}{r_{\psi_{\mathbf{u}}(1)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_1} \right) \times \left( \prod_{m=2}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi_{\mathbf{u}}(m)}}{r_{\psi_{\mathbf{u}}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_m} \right); I_k(n) \right) \end{aligned}$$

$$= E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \left( \left( \frac{\Phi_{\alpha}}{r_{\alpha}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_1} \right) \times \left( \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi \mathbf{u} - \mathbf{e}_{\alpha}(m)}}{r_{\psi \mathbf{u} - \mathbf{e}_{\alpha}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_{m+1}} \right); I_k(n) \right). \quad (15)$$

Here we express  $I_k(n_{k-1} \cdots n_0)$ ,  $n = n_{k-1} \cdots n_0$ , as

$$I_k(n_{k-1} \cdots n_0) = \bigcup_{0 \leq b_0, \dots, b_{\beta_1-k+1} \leq q-1} I_{\beta_1+2}(n_{k-1} \cdots n_0 b_{\beta_1-k+1} \cdots b_0). \quad (16)$$

Since  $\left( \frac{\Phi_{\alpha}}{r_{\alpha}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_1}$  in (15) is  $\mathcal{F}_{\beta_1+2}$ -measurable (see Lemma 2), it takes a constant value  $C_{b_{\beta_1-k+1} \cdots b_0}$  on  $I_{\beta_1+2}(n_{k-1} \cdots n_0 b_{\beta_1-k+1} \cdots b_0)$ . Hence, by (15) and (16),

$$\begin{aligned} & E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi \mathbf{u}(m)}}{r_{\psi \mathbf{u}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_m}; I_k(n) \right) \\ &= \sum_{0 \leq b_0, \dots, b_{\beta_1-k+1} \leq q-1} C_{b_{\beta_1-k+1} \cdots b_0} \\ & \quad \times E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi \mathbf{u} - \mathbf{e}_{\alpha}(m)}}{r_{\psi \mathbf{u} - \mathbf{e}_{\alpha}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_{m+1}}; I_{\beta_1+2}(n_{k-1} \cdots n_0 b_{\beta_1-k+1} \cdots b_0) \right). \end{aligned}$$

By repeating this  $|\mathbf{u}| - 1$  times, there exists an integer  $l$  with  $0 \leq l \leq q - 2$  such that  $E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi \mathbf{u}(m)}}{r_{\psi \mathbf{u}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_m}; I_k(n) \right)$  is a linear combination of  $E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_{|\mathbf{u}|}}; I_{\beta_{|\mathbf{u}|-1}+2}(n') \right)$  over  $n'$ 's. Therefore the assertion follows from Lemma 11.  $\square$

By Lemmas 11 and 12 with  $k = 1$ , we have easily the following.

**Lemma 13.** *For any  $\mathbf{u}$  with  $|\mathbf{u}| \geq 1$  and  $\{\beta_m\}_{m=1}^{|\mathbf{u}|}$  with  $0 \leq \beta_1 < \beta_2 < \cdots < \beta_{|\mathbf{u}|}$ , we have*

$$E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi \mathbf{u}(m)}}{r_{\psi \mathbf{u}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{\beta_m} \right) = 0.$$

**Proposition 4.** (i) *If  $\mathbf{u} = \mathbf{e}_l$ , then*

$$\mathcal{D}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l, k}(x) = \frac{1}{q} \sum_{j=0}^k \sum_{n=0}^{q^j-1} \mu_{\mathbf{d}, \mathbf{r}}(I_j(n)) \mathbf{1}_{I_j(n)}(x) \int_0^{\phi^j(x)} \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) d\mu_{\mathbf{r}_{\sigma^n}, \mathbf{r}}.$$

(ii) *If  $|\mathbf{u}| \geq 2$ , then*

$$\mathcal{D}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, k}(x) = \sum_{j=0}^{k-|\mathbf{u}|+1} \sum_{\substack{\alpha=0 \\ u_{\alpha}>0}}^{q-2} \left( \left( \frac{\Phi_{\alpha}}{r_{\alpha}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) \right)$$

$$\times \sum_{n=0}^{q^{j+1}-1} \mu_{\mathbf{d}, \mathbf{r}}(I_{j+1}(n)) \mathbf{1}_{I_{j+1}(n)}(x) \left( \mathcal{D}_{\mathbf{r}_{\sigma^n}, \mathbf{r}, \mathbf{u} - \mathbf{e}_\alpha, k-j-1} \circ \phi^{j+1}(x) \right).$$

*Proof.* Taking  $\mathbf{u} = \mathbf{e}_l$  in Definition 5, we have

$$\mathcal{D}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l, k}(x) = \frac{1}{q} \sum_{j=0}^k \int_0^x \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j d\mu_{\mathbf{d}, \mathbf{r}}.$$

If  $x \in I_j(n)$ , then, by Lemmas 11 and 6,

$$\begin{aligned} \int_0^x \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j d\mu_{\mathbf{d}, \mathbf{r}} &= \mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j; I_j(n) \cap [0, x] \right) \\ &= \mu_{\mathbf{d}, \mathbf{r}}(I_j(n)) \int_0^{\phi^j(x)} \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) d\mu_{\mathbf{r}_{\sigma^n}, \mathbf{r}}, \end{aligned}$$

which gives (i).

We express the sum  $\sum_{0 \leq i_1 < \dots < i_{|\mathbf{u}|} \leq k}$  in Definition 5 as  $\sum_{j=0}^{k-|\mathbf{u}|+1} \sum_{j+1 \leq i_2 < \dots < i_{|\mathbf{u}|} \leq k}$ , then, set  $i'_{m-1} = i_m - j - 1$ . Then we have, by (14),

$$\begin{aligned} \mathcal{D}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, k}(x) &= \frac{1}{q} \sum_{j=0}^{k-|\mathbf{u}|+1} \sum_{0 \leq i'_1 < \dots < i'_{|\mathbf{u}|-1} \leq k-j-1} \sum_{\psi_{\mathbf{u}}} \int_0^x \left( \left( \frac{\Phi_{\psi_{\mathbf{u}}(1)}}{r_{\psi_{\mathbf{u}}(1)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j \right) \times \left( \prod_{m=2}^{|\mathbf{u}|} \left( \frac{\Phi_{\psi_{\mathbf{u}}(m)}}{r_{\psi_{\mathbf{u}}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i'_{m-1}+j+1} \right) d\mu_{\mathbf{d}, \mathbf{r}} \\ &= \frac{1}{q} \sum_{j=0}^{k-|\mathbf{u}|+1} \sum_{0 \leq i'_1 < \dots < i'_{|\mathbf{u}|-1} \leq k-j-1} \sum_{\substack{\alpha=0 \\ u_\alpha > 0}}^{q-2} \sum_{\psi_{\mathbf{u}-\mathbf{e}_\alpha}} \int_0^x \left( \left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j \right) \times \left( \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi_{\mathbf{u}-\mathbf{e}_\alpha}(m)}}{r_{\psi_{\mathbf{u}-\mathbf{e}_\alpha}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i'_m+j+1} \right) d\mu_{\mathbf{d}, \mathbf{r}}. \end{aligned}$$

By Lemma 2,  $\left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j$  is  $\mathcal{F}_{j+2}$ -measurable. From (4) and Lemmas 11 and 12, it follows that

$$\mathbb{E}_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi_{\mathbf{u}-\mathbf{e}_\alpha}(m)}}{r_{\psi_{\mathbf{u}-\mathbf{e}_\alpha}(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i'_m+j+1} \middle| \mathcal{F}_{j+2} \right) = 0.$$

Hence we have, by Lemma 7,

$$\mathcal{D}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, k}(x) = \frac{1}{q} \sum_{j=0}^{k-|\mathbf{u}|+1} \sum_{0 \leq i'_1 < \dots < i'_{|\mathbf{u}|-1} \leq k-j-1} \sum_{\substack{\alpha=0 \\ u_\alpha > 0}}^{q-2} \sum_{\psi_{\mathbf{u}-\mathbf{e}_\alpha}} \left( \left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) \right)$$

$$\times E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}}{r_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i'_m} \circ \phi^{j+1}; [0, x] \right). \quad (17)$$

Lemmas 5 and 13 give, for every  $l$ ,

$$\begin{aligned} & E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}}{r_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i'_m} \circ \phi^{j+1}; I_{j+1}(l) \right) \\ &= \mu_{\mathbf{d}, \mathbf{r}}(I_{j+1}(l)) E_{\mu_{\mathbf{r}_{\sigma^l}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}}{r_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i'_m} \right) = 0. \end{aligned} \quad (18)$$

Lemma 6 gives, for  $x \in I_{j+1}(n)$ ,

$$\begin{aligned} & E_{\mu_{\mathbf{d}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}}{r_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i'_m} \circ \phi^{j+1}; I_{j+1}(n) \cap [0, x] \right) \\ &= \mu_{\mathbf{d}, \mathbf{r}}(I_{j+1}(n)) E_{\mu_{\mathbf{r}_{\sigma^n}, \mathbf{r}}} \left( \prod_{m=1}^{|\mathbf{u}|-1} \left( \frac{\Phi_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}}{r_{\psi \mathbf{u} - \mathbf{e}_\alpha(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{i'_m}; [0, \phi^{j+1}(x)] \right). \end{aligned} \quad (19)$$

Combining (17), (18), (19), and Definition 5, we obtain for  $x \in I_{j+1}(n)$

$$\begin{aligned} \mathcal{D}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, k}(x) &= \sum_{j=0}^{k-|\mathbf{u}|+1} \sum_{\substack{\alpha=0 \\ u_\alpha > 0}}^{q-2} \left( \left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) \right) \\ &\quad \times \mu_{\mathbf{d}, \mathbf{r}}(I_{j+1}(n)) \left( \mathcal{D}_{\mathbf{r}_{\sigma^n}, \mathbf{r}, \mathbf{u} - \mathbf{e}_\alpha, k-j-1} \circ \phi^{j+1}(x) \right), \end{aligned}$$

which gives (ii). □

## 6 Completion of the proof of Theorem 1

**Lemma 14.** *For any  $\mathbf{u}$  with  $|\mathbf{u}| \geq 1$ , we have*

$$\max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|} \sup_{\mathbf{d}'} \|\mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'}\|_\infty \leq \frac{(q-1)^{|\mathbf{u}|-1}}{q \max_{0 \leq a \leq q-1} \{r_a\}} \left( \frac{2}{\min_{0 \leq a \leq q-1} \{r_a\}} \frac{1}{1 - \max_{0 \leq a \leq q-1} \{r_a\}} \right)^{|\mathbf{u}|},$$

where  $\|\cdot\|_\infty$  means the supremum norm for  $x \in I$ .



*Proof.* Define  $\mathcal{M}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, j}(x)$  by

$$\mathcal{M}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, j}(x) = \begin{cases} \frac{1}{q} \sum_{n=0}^{q^j-1} \mu_{\mathbf{d}, \mathbf{r}}(I_j(n)) \mathbf{1}_{I_j(n)}(x) \int_0^{\phi^j(x)} \left| \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right| d\mu_{\mathbf{r}_{\sigma^n}, \mathbf{r}}, & \mathbf{u} = \mathbf{e}_l, \\ \sum_{\substack{\alpha=0 \\ u_\alpha > 0 \\ q^{j+1}-1}}^{q-2} \left| \left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) \right| \\ \times \sum_{n=0}^{q^{j+1}-1} \mu_{\mathbf{d}, \mathbf{r}}(I_{j+1}(n)) \mathbf{1}_{I_{j+1}(n)}(x) |\mathcal{T}_{\mathbf{r}_{\sigma^n}, \mathbf{r}, \mathbf{u}-\mathbf{e}_\alpha} \circ \phi^{j+1}(x)|, & |\mathbf{u}| \geq 2. \end{cases}$$

Since  $|\Phi_l(x)| \leq 1$ , we have for  $j \in \mathbf{N} \cup \{0\}$

$$\left| \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) \right| \leq \frac{2}{\min_{0 \leq a \leq q-1} \{r_a\}}. \quad (20)$$

Fix  $x \in I$ . For every  $j$ , there exists an  $m_j$  such that  $x \in I_j(m_j)$ . Then, by (20),

$$\begin{aligned} \mathcal{M}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l, j}(x) &\leq \frac{1}{q} \mu_{\mathbf{d}, \mathbf{r}}(I_j(m_j)) \frac{2}{\min_{0 \leq a \leq q-1} \{r_a\}} \mu_{\mathbf{r}_{\sigma^{m_j}}, \mathbf{r}}([0, \phi^j(x)]) \\ &\leq \frac{2}{q \min_{0 \leq a \leq q-1} \{r_a\}} \left( \max_{0 \leq a \leq q-1} \{d_a\} \right) \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^{j-1} \\ &\leq \frac{2}{q \min_{0 \leq a \leq q-1} \{r_a\}} \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^{j-1}. \end{aligned}$$

Hence

$$\sum_{j=0}^{\infty} \|\mathcal{M}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l, j}\|_{\infty} \leq \frac{2}{q \min_{0 \leq a \leq q-1} \{r_a\}} \frac{\left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^{-1}}{1 - \max_{0 \leq a \leq q-1} \{r_a\}}. \quad (21)$$

From  $\|\mathcal{T}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l}\|_{\infty} \leq \sum_{j=0}^{\infty} \|\mathcal{M}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l, j}\|_{\infty}$  and (21), it follows that

$$\sup_{\mathbf{d}'} \|\mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{e}_l}\|_{\infty} \leq \frac{2}{q \min_{0 \leq a \leq q-1} \{r_a\}} \frac{\left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^{-1}}{1 - \max_{0 \leq a \leq q-1} \{r_a\}}. \quad (22)$$

Fix  $x \in I$ . For every  $j$ , there exists an  $m_j$  such that  $x \in I_{j+1}(m_j)$ . Then, by (20), we have for  $|\mathbf{u}| \geq 2$

$$\mathcal{M}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, j}(x) \leq \frac{2}{\min_{0 \leq a \leq q-1} \{r_a\}} \mu_{\mathbf{d}, \mathbf{r}}(I_{j+1}(m_j)) \sum_{\substack{\alpha=0 \\ u_\alpha > 0}}^{q-2} |\mathcal{T}_{\mathbf{r}_{\sigma^{m_j}}, \mathbf{r}, \mathbf{u}-\mathbf{e}_\alpha} \circ \phi^{j+1}(x)|$$

$$\begin{aligned}
&\leq \frac{2(q-1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \left( \max_{0 \leq a \leq q-1} \{d_a\} \right) \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j \max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|-1} |\mathcal{T}_{\mathbf{r}_{\sigma^{m_j}, \mathbf{r}, \mathbf{u}'}} \circ \phi^{j+1}(x)| \\
&\leq \frac{2(q-1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j \max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|-1} \sup_{\mathbf{d}'} \|\mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'}\|_{\infty}.
\end{aligned}$$

Hence

$$\sum_{j=0}^{\infty} \|\mathcal{M}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, j}\|_{\infty} \leq \frac{2(q-1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \frac{1}{1 - \max_{0 \leq a \leq q-1} \{r_a\}} \max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|-1} \sup_{\mathbf{d}'} \|\mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'}\|_{\infty}. \quad (23)$$

From  $\|\mathcal{T}_{\mathbf{d}, \mathbf{r}, \mathbf{u}}\|_{\infty} \leq \sum_{j=0}^{\infty} \|\mathcal{M}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, j}\|_{\infty}$  and (23), it follows that

$$\max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|} \sup_{\mathbf{d}'} \|\mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'}\|_{\infty} \leq \frac{2(q-1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \frac{1}{1 - \max_{0 \leq a \leq q-1} \{r_a\}} \max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|-1} \sup_{\mathbf{d}'} \|\mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'}\|_{\infty}.$$

By repeating this  $|\mathbf{u}| - 1$  times, there exists an integer  $l$  with  $0 \leq l \leq q - 1$  such that

$$\max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|} \sup_{\mathbf{d}'} \|\mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'}\|_{\infty} \leq \left( \frac{2(q-1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \frac{1}{1 - \max_{0 \leq a \leq q-1} \{r_a\}} \right)^{|\mathbf{u}|-1} \sup_{\mathbf{d}'} \|\mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{e}_l}\|_{\infty}. \quad (24)$$

Combining (24) with (22), we obtain the assertion.  $\square$

**Lemma 15.** *For any  $\mathbf{u}$  with  $|\mathbf{u}| \geq 1$ , we have*

$$\max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|} \sup_{\mathbf{d}'} \left\| \mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'} - \mathcal{D}_{\mathbf{d}', \mathbf{r}, \mathbf{u}', k} \right\|_{\infty} \leq P_{|\mathbf{u}|-1}(k) \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^k,$$

where  $P_{|\mathbf{u}|-1}(k)$  is a polynomial of  $k$  with degree  $|\mathbf{u}| - 1$ .

*Proof.* We prove this by induction on  $|\mathbf{u}|$ . By the same argument as in the proof of Lemma 14,

$$|\mathcal{T}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l}(x) - \mathcal{D}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l, k}(x)| \leq \sum_{j=k+1}^{\infty} |\mathcal{M}_{\mathbf{d}, \mathbf{r}, \mathbf{e}_l, j}(x)| \leq \frac{2}{q \min_{0 \leq a \leq q-1} \{r_a\}} \frac{\left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^k}{1 - \max_{0 \leq a \leq q-1} \{r_a\}}.$$

Hence

$$\sup_{\mathbf{d}'} \|\mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{e}_l} - \mathcal{D}_{\mathbf{d}', \mathbf{r}, \mathbf{e}_l, k}\|_{\infty} \leq \frac{2}{q \min_{0 \leq a \leq q-1} \{r_a\}} \frac{\left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^k}{1 - \max_{0 \leq a \leq q-1} \{r_a\}}.$$

Fix  $x \in I$ . For every  $j$ , there exists an  $m_j$  such that  $x \in I_{j+1}(m_j)$ . Then, by (20), we have for  $|\mathbf{u}| \geq 2$

$$\begin{aligned}
& |\mathcal{T}_{\mathbf{d}, \mathbf{r}, \mathbf{u}}(x) - \mathcal{D}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, k}(x)| \\
& \leq \sum_{j=k-|\mathbf{u}|+2}^{\infty} \sum_{\substack{\alpha=0 \\ u_\alpha > 0}}^{q-2} \left| \left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) \right| \\
& \quad \times \sum_{n=0}^{q^{j+1}-1} \mu_{\mathbf{d}, \mathbf{r}}(I_{j+1}(n)) \mathbf{1}_{I_{j+1}(n)}(x) \left| \mathcal{T}_{\mathbf{r}_{\sigma^n}, \mathbf{r}, \mathbf{u} - \mathbf{e}_\alpha} \circ \phi^{j+1}(x) \right| \\
& + \sum_{j=0}^{k-|\mathbf{u}|+1} \sum_{\substack{\alpha=0 \\ u_\alpha > 0}}^{q-2} \left| \left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) \right| \\
& \quad \times \sum_{n=0}^{q^{j+1}-1} \mu_{\mathbf{d}, \mathbf{r}}(I_{j+1}(n)) \mathbf{1}_{I_{j+1}(n)}(x) \left| \mathcal{T}_{\mathbf{r}_{\sigma^n}, \mathbf{r}, \mathbf{u} - \mathbf{e}_\alpha} \circ \phi^{j+1}(x) - \mathcal{D}_{\mathbf{r}_{\sigma^n}, \mathbf{r}, \mathbf{u} - \mathbf{e}_\alpha, k-j-1} \circ \phi^{j+1}(x) \right| \\
& \leq \sum_{j=k-|\mathbf{u}|+2}^{\infty} \frac{2}{\min_{0 \leq a \leq q-1} \{r_a\}} \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j \sum_{\substack{\alpha=0 \\ u_\alpha > 0}}^{q-2} \left\| \mathcal{T}_{\mathbf{r}_{\sigma^{m_j}}, \mathbf{r}, \mathbf{u} - \mathbf{e}_\alpha} \right\|_\infty \\
& + \sum_{j=0}^{k-|\mathbf{u}|+1} \frac{2}{\min_{0 \leq a \leq q-1} \{r_a\}} \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j \sum_{\substack{\alpha=0 \\ u_\alpha > 0}}^{q-2} \left\| \mathcal{T}_{\mathbf{r}_{\sigma^{m_j}}, \mathbf{r}, \mathbf{u} - \mathbf{e}_\alpha} - \mathcal{D}_{\mathbf{r}_{\sigma^{m_j}}, \mathbf{r}, \mathbf{u} - \mathbf{e}_\alpha, k-j-1} \right\|_\infty \\
& \leq \frac{2(q-1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \frac{\left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^{k-|\mathbf{u}|+2}}{1 - \max_{0 \leq a \leq q-1} \{r_a\}} \max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|-1} \sup_{\mathbf{d}'} \left\| \mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'} \right\|_\infty \\
& + \frac{2(q-1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \sum_{j=0}^{k-|\mathbf{u}|+1} \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j \max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|-1} \sup_{\mathbf{d}'} \left\| \mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'} - \mathcal{D}_{\mathbf{d}', \mathbf{r}, \mathbf{u}', k-j-1} \right\|_\infty.
\end{aligned}$$

Hence, by Lemma 14 and the assumption of induction,

$$\begin{aligned}
& \max_{\mathbf{u}', |\mathbf{u}'|=|\mathbf{u}|} \sup_{\mathbf{d}'} \left\| \mathcal{T}_{\mathbf{d}', \mathbf{r}, \mathbf{u}'} - \mathcal{D}_{\mathbf{d}', \mathbf{r}, \mathbf{u}', k} \right\|_\infty \\
& \ll \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^{k-|\mathbf{u}|+2} + \sum_{j=0}^{k-|\mathbf{u}|+1} \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j P_{|\mathbf{u}|-2}(k-j-1) \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^{k-j-1} \\
& = \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^{k-|\mathbf{u}|+2} + \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^{k-1} \sum_{j=0}^{k-|\mathbf{u}|+1} P_{|\mathbf{u}|-2}(k-j-1),
\end{aligned}$$

where the implied constant depends only on  $q$ ,  $\mathbf{r}$ , and  $|\mathbf{u}|$ . Since  $\sum_{j=0}^{k-|\mathbf{u}|+1} P_{|\mathbf{u}|-2}(k-j-1)$  is a polynomial of  $k$  with degree  $|\mathbf{u}| - 1$ , we obtain the assertion.  $\square$

By Lemma 15, we see that

$$\lim_{k \rightarrow \infty} \mathcal{D}_{\mathbf{d}, \mathbf{r}, \mathbf{u}, k}(x) = \mathcal{T}_{\mathbf{d}, \mathbf{r}, \mathbf{u}}(x)$$

holds uniformly for  $x \in I$ . Thus we obtain Theorem 1 by Propositions 3 and 4.  $\square$

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